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By Robert Kappus

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SUMMARY

The result of the present investigations is substantially as follows. The buckling of centrally loaded columns of open section is always accompanied by a twist if the cross section discloses neither axial nor point symmetry. Then the cross sections twist about an axis of rotation D, the location of which depends upon the shape of the median line of the section, the wall thickness and column length, and the limiting conditions. There are three such axes and consequently three different critical compressive stresses (twisting failure stresses). With point symmetry of the cross section (wherein the case of double-column symmetry is contained as a special case) the three critical compressive stresses are given in two Euler stresses for (twist-free) buckling in direction of the two principal axes of inertia and one twisting failure stress for twisting about an axis of rotation passing through the center of gravity. With simple cross-section symmetry, it finally affords one Euler stress for buckling in direction of the axis of symmetry and two twisting failure stresses for twisting about two axes of rotation in the plane of symmetry. Buckling perpendicular to the axis of symmetry is therefore connected with a twist of the column. The thicker the wall and the greater the length of the columns the more the effect of the twist is neutralized, as the distance between center of rotation and center of gravity continues to increase until finally, the observed buckling is practically free from twist.

This holds for symmetrical as well as for unsymmetrical sections; the Euler formula gives, in this case, good (slightly too high) approximate values.

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*"Drillknicken zentrisch gedrückter Stäbe mit offenem Profil im elastischen Bereich." Luftfahrtforschung, vol. 14, no. 9, September 20, 1937, pp. 444-457.

INTRODUCTION

For centrally loaded columns of solid cross section or thick-walled, hollow cross section, only one type of instability of the "straight" equilibrium pattern is recognized - namely, twist-free buckling, which is usually termed flexural buckling or (in the elastic range) Euler buckling. For cross sections of thin-walled open or closed section and short column length, another type of instability in the original equilibrium pattern is of importance, namely, the buckling of the less bending-resistant walls due to inferior wall thickness. Lastly, there is yet a third instability phenomenon observed on thin-walled columns of open section - that is, torsional buckling or, as it is also called, twisting failure. Here, under a certain critical compressive force, the straight equilibrium pattern may be accompanied by infinitely adjacent twisted equilibrium patterns whereby, in contrast to the case of buckling, the cross-sectional shape is preserved and the strains due to reciprocal twisting of the cross sections about a well-defined axis of rotation are given.

The reason for the importance of twisting failure being restricted to columns of thin-walled open section, is due to the fact that open sections - especially, when thin-walled - possess an extremely low twisting strength. Twisting failure is most often and most clearly observed in airplane designs where the employed sections usually have much thinner walls than is customary on other structures. Aviation literature on this stability problem is substantially exhausted with a theoretical treatise by H. Wagner (reference 1), and a further report by H. Wagner and W. Pretschner (reference 2) which also contains test data for angle sections. In structural engineering literature, the twisting failure of centrally loaded columns does not appear at all - as far as the writer known - except for one recent article by H. and F. Bleich (reference 3). In this report the writers use the energy method for the derivation of the differential equations for the strain quantities, while Wagner attains a clearer differential equation for the angle of twist from a consideration of the moment equilibrium about the column axis. In spite of the fundamentally identical assumptions, the results do not agree, and for the following reasons:

Wagner defines the axis of rotation, to begin with, on the inconclusive assumption that the center of rotation on twisting failure is coincident with the so-called "center of shear," while H. and F. Bleich's formula for the work of the external forces, fails to include the term which accounts for the twisting of the cross sections.

In the following a complete theory of twisting failure by the energy method is developed, based on substantially the same assumptions as those employed by Wagner and Bleich. Problems treated in detail are: the stress and strain condition under St. Venant twist and in twist with axial constraint; the concept of shear center and the energy method for problems of elastic stability.

1. CENTER OF ROTATION AND WARPING UNDER ST. VENANT TWIST; SHEAR CENTER M, TWISTING WITH AXIAL CONSTRAINT

Figure 1 shows the median line of an open section with several essential symbols. A rectangular system of coordinates x, y, z is passed through the centroid S of one end cross section of the correlated column; x, y are arbitrary centroidal axes of the cross section, and z the column axis; axes x, y, z are to form a right-hand system; i.e., the z axis in figure 1 points toward the observer. We arbitrarily fix a direction of rotation and thereby allocate to each point of the section center line a circumferential coordinate u. Suppose the direction of the positive sense of rotation indicates the positive tangential direction t; at right angles to it, the positive normal direction n points toward the right, as observed when looking in t direction. The distances r_t and r_n of the tangent and of the normal, equal the distance r of the particular point from the centroid S. A line drawn from S in positive n direction indicates the positive r_t direction. This and the x direction form an angle α to be measured in the positive sense of rotation (rotating from $+x$ toward $+y$). If the t direction relative to S has, say, a positive sense of rotation, r_t is positive according to the preceding notation. Correspondingly, r_n is counted positive when n rotates positive in relation to S.

Consider a column, as in figure 1, under the effect of twisting moments T applied at its two free end cross sections in conformity with St. Venant's theory of twist. The cross-sectional shape is preserved and all stresses and strains other than the angle of twist φ are unaffected by z. The "referred" angle of twist $\varphi' = \frac{d\varphi}{dz}$ is constant. With $G = \frac{E}{2(1+\mu)}$ as shear modulus and $J_T = \left(\frac{1}{3}\right) U \cdot s^3$ as twisting strength[1)] whereby U is the developed contour of the center line of the section and s is the wall thickness, it is $T = G\, J_T\, \varphi'$. The linear distribution of the shearing (twisting) stresses τ_T over the wall thickness in any cross section is exactly the same as in a small rectangular strip. In particular, the shearing stress in the center line of the section is zero - as a result of which there is no angular change between it and the elements of the surface (longitudinal fibers). Every element of the median line of the section remains perpendicular to the correlated fiber, which remains straight, according to the linearized theory. Then since the fibers, depending on their distance from the center of rotation, are differently inclined, the cross section does not remain flat. If the rotation is, for instance, about an axis passing through S, the displacement of a point in the cross-sectional plane is given with $V = r\,\varphi$ (fig. 2); the component in tangent direction is accordingly:

$$V_t = r_t\,\varphi \tag{1}$$

There being no angular change between the circumferential and the fiber element, the displacement is:

$$\gamma = \frac{\partial V_t}{\partial z} + \frac{\partial W}{\partial u} = 0 \tag{2}$$

where W is the cross-sectional warping (positive in direction of the positive z axis). With the introduction of the unit warping w (of the dimension of an area, fig. 1) as

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1)For variable wall thickness $J_T = \frac{1}{3} \int_{u=u_0}^{u=u_R} s^3\, du.$

$$W = -\varphi' w \tag{3}$$

equation (2) (with equation 1) gives - while observing that according to assumption, W, φ', w are to be independent of z, $\frac{dw}{du} = r_t$,[2] that is:

$$w = w_o + \int_o^u r_t \, du \tag{4}$$

If the rotation instead of about S is around any other point of rotation D with the coordinates x_D and y_D, through which a coordinate system $\tilde{x}$, $\tilde{y}$ is placed parallel to x, y, it correspondingly gives

$$\tilde{V}_t = \tilde{r}_t \varphi \tag{5}$$

$$\tilde{W} = -\varphi' \tilde{w} \tag{6}$$

and

$$\tilde{w} = \tilde{w}_o + \int_o^u \tilde{r}_t \, du \tag{7}$$

Figure 2 illustrates the following relations:

$$\tilde{r}_t = r_t + \tilde{x}_s \cos \alpha + \tilde{y}_s \sin \alpha \tag{8}$$

and

$$du \cos \alpha = dy, \quad du \sin \alpha = - dx$$

hence

$$\tilde{w} - \tilde{w}_o = w - w_o + \tilde{x}_s(y-y_o) - \tilde{y}_s(x-x_o)$$

i.e.,

$$\tilde{w} = w + \tilde{x}_s \, y - \tilde{y}_s \, x + K \tag{9}$$

or

$$\tilde{w} = w - x_D \, y + y_D \, x + K \tag{9'}$$

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[2] Strictly speaking, these relations are valid for the center line only; $w = \int_o^u r_t \, du + r_n \, n$ is more exact. Since n, at the most, is equal to $s/2$, the cross-sectional warping $r_n \, n$, superposing themselves on the circumferential warping $\int_o^u r_t \, du$, can be neglected.

The unit warping $\tilde{w}$ and w for one and the same section differ accordingly by one linear expression in x and y. In both cases the same stress condition prevails and the reciprocal strains are completely the same. The difference in warping lies merely in the relation to different reference planes, which are perpendicular to the momentary axes of rotation of the twisted column. According to St. Venant's theory of twist, the position of the axis of rotation is, on principle, undetermined because, since all fibers remain straight, each fiber and each one parallel to it, may serve as axis of rotation. Fixing the axis of rotation - say, through hinges - and prescribing the warping of any point, establishes the warping of all points if the load in the end sections is applied in accord with the theory. But the axis of rotation can also be established, as seen from equation (9'), by prescribing the warping of three (not lying on one straight line) points arbitrarily, because three of such equations suffice for the correct solution of the unknown factors K, x_D, and y_D. The frequently entertained opinion that in St. Venant's twist the axis of rotation would always have to pass through one definite point, the center of shear M, is therefore untenable.

Since the concept of shear center is being used repeatedly in this report, a brief discussion of the formulas defining its position is given. Putting a thin-walled, open-section column under transverse load results, in general, in twisting in addition to warping. Warping is not accompanied by twist (definition of shear center) if the transverse force passes through the center of shear M. Assuming linear distribution of the bending stresses in twist-free bending and defining the shearing stresses due to transverse force in the usual manner from equilibrium conditions, the statement that in this case the transverse force relative to the shear center may have no moment, leads to the subsequent definition equations for the shear center:[3)]

$$\int_F x\, w^*\, dF = 0 \quad \text{and} \quad \int_F y\, w^*\, dF = 0 \qquad (10)$$

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3)For the derivation of these formulas as well as of those used for integration of arbitrary sections, see the article entitled: "Shear Center of Thin-Walled Sections," by W. Lücker, published by the Static Test Branch of the D.V.L.

Here x, y denote any coordinate system passing through S, and the unit warpings w^* referred to shear center M are given with

$$w^* = w_o^* + \int_o^u r_t^* \, du \tag{11}$$

whereby r_t^* equals the distance of the circumferential tangent from the shear center (fig. 2). On the basis of the coordinate system $\bar{x}, \bar{y}$ parallel to x, y placed through the arbitrary (to be chosen properly) reference point O, figure 2 indicates

$$\bar{r}_t = r_t^* + \bar{x}_M \cos \alpha + \bar{y}_M \sin \alpha \tag{12}$$

From

$$\bar{w} = \bar{w}_o + \int_o^u \bar{r}_t \, du \tag{13}$$

follows

$$w^* = \bar{w} - \bar{x}_M y + \bar{y}_M x + K \tag{14}$$

The introduction of (14) in (10) then gives the two equations (15) for the coordinates $\bar{x}_M \bar{y}_M$ of the shear center M

$$\left.\begin{aligned} \bar{x}_M J_x - \bar{y}_M J_{xy} &= \int_F y \bar{w} \, dF \\ - \bar{x}_M J_{xy} + \bar{y}_M J_y &= - \int_F x \bar{w} \, dF \end{aligned}\right\} \text{------} \tag{15}$$

While no particular importance attaches to M in the case of St. Venant's twist, it plays a significant part in twist with axial constraint, where $\varphi' = d\varphi/dz$ is no longer constant. This case occurs, for example, if the column is clamped at one end or when, other than the end moments, individual moments or distributed moments (m_z) are applied in addition. Then the axis of rotation always passes through the shear center as shown later (equation 18). All other fibers bend, because the warping W^* is not constant as a result of the changeability of φ' and strains $\epsilon^* = \partial W^*/\partial z$ and stresses $\sigma^* = E\epsilon^*$ occur in consequence in column direction. Inasmuch as the stresses σ^* themselves then are variable again, it simultaneously results, for reasons of equilibrium, in shearing stresses τ^* constantly distributed over the wall thickness, coursing in the direction of the column axis and of the circumference.

Unless the column length is abnormally small, these shearing stresses τ^* remain small compared to the axial stresses σ^*, so that the related equations γ, in accordance with the usual assumption of the elementary beam theory, may be ignored. The warping W^* of open sections is, therefore, under the assumption of preservation of cross-sectional shape, to be computed in the same manner as in pure St. Venant's twist, where no slippage in the section line occurs either. From the partial differential equation (2) circumscribed for V_t^* and W^* in conjunction with the unit warping w^* defined by equation (11) follows:

$$W^* = -\varphi' w^* + f(z)$$

hence

$$\sigma^* = E\,\epsilon^* = E\,\frac{\partial W^*}{\partial z} = -E(\varphi'' w^* + f'(z)) \tag{17}$$

As the twisted column is to transmit neither axial load nor bending moments, the axial stresses σ^* for each cross section must form an equilibrium group; that is, if F is the cross-sectional area

$$\int_F \sigma^*\,dF = 0, \quad \int x\,\sigma^*\,dF = 0, \quad \int y\,\sigma^*\,dF = 0 \tag{18a,b,c}$$

Since $f'(z)$ is constant over F, and x and y are centroidal axes, the equations (10) follow direct from (18b,c), which proves the coincidence of center of rotation and center of shear in this case. Equation (18a) gives:

$$f'(z) = \varphi''\,\frac{1}{F}\int_F w^*\,dF \tag{19}$$

Compliance with

$$\int_F w^*\,dF = 0 \tag{20}$$

which is always possible with suitable choice of w_0^* in (11) or of K in (14), affords the following simple relations:

$$W^* = -\varphi' w^* \tag{16'}$$

and

$$\sigma^* = -E\,\varphi'' w^* \tag{17'}$$

The shearing stresses τ^* follow from the equilibrium condition for forces in the z direction at a small element s du dz:

$$\frac{\partial(\tau^* s)}{\partial u} + s\,\frac{\partial\sigma^*}{\partial z} = 0 \qquad (21)$$

The shearing stresses τ^* dF form a force couple of the magnitude $- E\,C^*\,\varphi'''$, whereby the term

$$C^* = \int_F w^{*2}\,dF \qquad (22)$$

is designated as warping strength relative to M.[4] The twisting moment of the shearing stresses τ_T being given with $G\,J_T\,\varphi'$, the term for the total twisting moment reads:

$$T = G\,J_T\,\varphi' - E\,C^*\,\varphi''' \qquad (23)$$

Despite the fact that the shearing stresses τ^* are usually substantially smaller than the shearing stresses τ_T, both share approximately alike on the total twisting moment, according to numerical calculations. This is due to the fact that the lever arms of τ^* are of the order of magnitude of the section dimensions, while the lever arm of τ_T is equal to 2/3 of the wall thickness.[5]

The calculation of the angle of twist along the z axis for predetermined individual twisting moments M_z and distributed moments m_z hinges on the integration of the differential equation:

$$-\frac{dT}{dz} = E\,C^*\,\varphi^{IV} - G\,J_T\,\varphi'' = m_z \qquad (24)$$

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[4] Since twisting with axial constraint is at times termed "flexural torsion," the quantity C^* is also designated as <u>torsion-bending</u> constant (C_{bd} or C_{BT} (references 1 and 2). But in the above case, the term "torsion bending" is misleading since twisting moments, but no bending moments, are transmitted (equations 18b,c).

[5] This is readily proved on a simple example of the I section. The axial constraint causes the flanges to warp and act like individual bars; that is, have linear distribution of σ^* and parabolic distribution of τ^*, while the web discloses no further stresses aside from τ_T.

The limiting and transition conditions depend upon the warping and stresses at the borders; they are obtained from (16'), (17'), and (23). With complete constraint, for example, we have $\varphi' = 0$; with no constraint, it is $\varphi'' = 0$.

2. STRESS AND STRAIN CONDITIONS AT TWISTING FAILURE; ENERGY TERMS A_1 AND A_a; DIFFERENTIAL EQUATIONS FOR THE STRAIN QUANTITIES ξ_s, η_s, φ

To gain an insight into the forces involved at twisting failure, consider a centrally loaded open-section column (fig. 1) under the effect of the critical compressive force P. In nontwisted condition each fiber is under the same compressive stress; that is, in the critical case the twisting-failure stress is $\sigma_D = P/F$. While σ_D and, consequently, the compression of the column represent finite quantities, the stresses and strains incurred on transition from the <u>straight</u> to the twisted equilibrium position, must be looked upon as small quantities of the first order of smallness. To them, the same relations, expressed in the preceding chapter with equations (16'), (17'), (18a), and (19) to (23) apply, except that now the temporarily unknown center of rotation D is no longer coincident with the shear center M, since the statement contained in (18b, 18c) no longer holds true. The torque of the shearing forces in each cross section with respect to the axis of rotation D can be computed from (23), where D substitutes for M; accordingly, instead of C^*

$$\tilde{C} = \int_F \tilde{w}^2 \, dF \tag{25}$$

whereby (7) and (9), respectively, are valid for the unit warping $\tilde{w}$ referred to center of rotation D, and $\tilde{w}_0$ and K must be so defined that

$$\int_F \tilde{w} \, dF = 0$$

For reasons of equilibrium, an anti-moment related to the external load is necessary; this is accomplished through the slope of the longitudinal fibers toward the axis of rotation at angle $\tilde{r}\,\varphi'$, so that the components of the compressive forces σ_D dF perpendicular to the axis of rotation, yield a rotating moment of

$$\int_F \sigma_D \, dF \, \tilde{r} \, \varphi' \, \tilde{r} = \varphi' \, \sigma_D \, \tilde{J}_p$$

about the column axis. The equilibrium of the moments about the axis of rotation then affords the differential equation:

$$E \, \tilde{C} \, \varphi^{IV} + (\sigma_D \, \tilde{J}_p - G \, J_T) \varphi'' = 0 \qquad (26)$$

from which with

$$\varphi = \tilde{a} \sin \frac{\pi}{l} z \qquad (27)$$

follows the critical stress as:

$$\sigma_D = \frac{G \, J_T + \frac{\pi^2}{l^2} E \, \tilde{C}}{\tilde{J}_p} \qquad (28)$$

But this formula does not as yet allow the calculation of the critical stress σ_D, since the position of D remains unknown. For the most general case of lack of cross-section symmetry, the two other conditions for moment equilibrium about two cross-sectional axes are still needed. In twist with axial constraint, these equilibrium conditions lead exactly to the shear center (equations (18b,c)); here, however, the $\tilde{\sigma}$ substituting for σ^* must be so distributed that their moments on the assumedly isolated part, are in equilibrium with the moments of the external forces due to the strain. Both moments are of the first order of smallness - the first on account of $\tilde{\sigma}$, the other on account of the lever arms. In Wagner's formula, where D is equated to M, that is, the same as (28) if D is replaced by M (that is, $\tilde{C}$ by C^* and $\tilde{J}_p$ by J_p^*), these equilibrium conditions are violated.

Intending to resume these equilibrium conditions in section 5, we now proceed in a simple manner to the derivation of the differential equation of twisting failure by means of the energy method. To this the energy of the internal and external forces (that is, the total potential) must be expressed through the strains, after which the differential equations - the mathematical expression for consistency of stress and strain on the element - then follow purely formally in accord with the rules of calculus of variations.

By A_i is meant the difference between the strain energy of the twisted and the straight equilibrium patterns under critical load P, and by A_a, the energy of the critical compressive force P on transition from straight to twisted equilibrium position. Both expressions must - since this relates to a stability problem - be quadratic functions of the strain quantities and, consequently, of the second order of smallness. Since the twisting process takes place of itself, i.e., without energy input, the total potential of the internal and external forces $A_i - A_a$ retains the value 0 of the initial condition. Of course, the validity of $A_i - A_a = 0$ is contingent upon the correct strain in the energy terms. For all other geometrically possible strains below the critical load, the energy required is $A_i - A_a > 0$, the very fact upon which the nonappearance of those strains is based. For actually possible strains, i.e., compatible with the equilibrium conditions, $A_i - A_a$ is a minimum of value 0. This dual statement supplies the critical load - the so-called stability limit, as well as the strains (defined up to an arbitrary factor). Expressed in the form of the calculus of variations, the minimum requirement reads $\delta(A_i - A_a) = 0$. It always embodies the other dictum $A_i - A_a = 0$, as can be proved.

The strain energy A_i absorbed by the column of length l at twisting failure, is:

$$A_i = \frac{1}{2} \int_{z=0}^{z=l} \left\{ E \int_F \tilde{\epsilon}^2 \, dF + G \, J_T \, \varphi'^2 \right\} dz \qquad (29)$$

The assumptions we shall make are as those made for twisting with axial constraint. The cross section retains its shape,[6] so that the strain may be expressed by three quan-

[6] No flexural loads being applied perpendicular to the surface of the column, the assumption of preservation of cross-sectional shape holds with great accuracy, failing only when the lengths or wall thicknesses are very small, as then the walls may buckle.

ties: For example, the angle of twist φ and the displacements ξ_s and η_s of the centroid S relative to the position of the center of rotation D, which are to be measured in the direction of the x and y axes of the cross sections of the untwisted column. Then

$$\tilde{W} = - \varphi' \tilde{w} \tag{30}$$

$$\tilde{\epsilon} = \frac{\partial \tilde{W}}{\partial z} = - \varphi'' \tilde{w} \tag{31}$$

replace (16'), (17'), and (20)

and

$$\int_F \tilde{w} \, dF = 0 \tag{32}$$

whereby (32) voices the equilibrium condition essential to the stability problem, namely, that the axial force (-P) remains unchanged. Equation (9) herewith becomes:

$$\tilde{w} = w + \tilde{x}_s y - \tilde{y}_s x - \frac{1}{F} \int_F w \, dF \tag{33}$$

with

$$\xi_s = - \tilde{y}_s \varphi, \quad \eta_s = + \tilde{x}_s \varphi \tag{34}$$

then follows from (31) the important relation for the strain distribution:

$$\tilde{\epsilon} = - \xi_s'' x - \eta_s'' y - \varphi'' \left(w - \frac{1}{F} \int_F w \, dF \right) \tag{35}$$

after which a simple computation and the introduction of (35) in (29) gives the strain energy A_i:

$$\begin{aligned} A_i = \frac{1}{2} \int_{z=0}^{z=l} \Big\{ & E J_y \xi_s''^2 + 2E J_{xy} \xi_s'' \eta_s'' + E J_x \eta_s''^2 + \\ & + 2E R_y \xi_s'' \varphi'' + 2E R_x \eta_s'' \varphi'' + \\ & + E C \varphi''^2 + G J_T \varphi'^2 \Big\} dz \end{aligned} \tag{36}$$

Herein:

$$R_y = \int_F x\ w\ dF \tag{37}$$

$$R_x = \int_F y\ w\ dF \tag{38}$$

$$C = \int_F w^2\ dF - \frac{1}{F}\left(\int_F w\ dF\right)^2 \tag{39}$$

The quantities R_y, R_x, and C are dependent on the shape of the cross section only. In conformity with the concept of static moment R_x (and R_y) are termed "curving moment for the x axis" (for the y axis, respectively), C the "curving strength relative to S" (cf. C^*, equation 22). For symmetrical sections, for instance, the course of the unit warping w is antisymmetrical, if the point of the section center line which meets the axis of symmetry is coordinated to zero warping (equation 4). Thus, $R_y = 0$ for symmetry with the x axis, and $R_x = 0$ for symmetry with the y axis. As x, y are centroidal axes, these relations are applicable to any other determination of the arbitrary constant w_o as well.

The energy A_a of the external forces - that is, of the compressive force P - due to contraction of chord on buckling of the fibers, is computed as for the buckling Euler column, since as a result of the rotation about the axis of rotation D all originally straight fibers change to plane curves. With ξ and η as displacement components of any cross section point, it is:

$$A_a = \sigma_D\ \frac{1}{2} \int_{z=o}^{z=l} \int_F \left(\xi'^2 + \eta'^2\right)\ dF\ dz \tag{40}$$

According to Euler's theorem of kinematics, the motion of a point is built up from the motion of any reference point and the rotary motion around that point. For S as reference point, it then is:

$$\xi = \xi_s - y\ \varphi, \quad \eta = \eta_s + x\ \varphi \tag{41}$$

Consequently, since x, y are centroidal axes,

$$A_a = \sigma_D \frac{1}{2} \int_{z=0}^{z=l} \left\{ F(\xi_s'^2 + \eta_s'^2) + (J_x + J_y)\varphi'^2 \right\} dz \qquad (42)$$

According to the rules of calculus of variations:

$$\delta \; (A_1 - A_a) = 0 \qquad (43)$$

then affords in simple manner the three differential equations for the strain quantities ξ_s, η_s, φ:

$$\underline{E\, J_y\, \xi_s^{IV}} + E\, \cancel{J_{xy}}\, \eta_s^{IV} + E\, \cancel{B_y}\, \varphi^{IV} + \underline{\sigma_D\, F\, \xi_s''} = 0 \qquad (44a)$$

$$E\, \cancel{J_{xy}}\, \xi_s^{IV} + \underline{E\, J_x\, \eta_s^{IV}} + E\, \cancel{B_x}\, \varphi^{IV} + \underline{\sigma_D\, F\, \eta_s''} = 0 \qquad (44b)$$

$$E\, \cancel{B_y}\, \xi_s^{IV} + E\, \cancel{B_x}\, \eta_s^{IV} + \underline{E\, C\, \varphi^{IV}} + \underline{(\sigma_D J_p - G\, J_T)\varphi''} = 0 \qquad (44c)$$

They state that the forces following from (30) must be in equilibrium at each column element (six equilibrium conditions in similar manner as this is expressed for a Euler column buckling in the xz plane by the differential equation $E\, J_y\, \xi^{IV} + P\, \xi'' = 0$, which, with observance of the strain law $B_y = - E\, J_y\, \xi''$ follows from the equilibrium conditions $\frac{\partial B_y}{\partial z} = Q_x$ and $\frac{\partial Q_x}{\partial z} = P\, \xi''$.

Because equations (44) are coupled together, the three quantities ξ_s, η_s, and φ usually occur concurrently; but if symmetry prevails, it is otherwise. With symmetry to the x axis, for instance, equation (44a) is independent of (44b) and (44c), because $J_{xy} = 0$ and $B_y = 0$. In this case a deflection ξ_s, independent of η_s and φ is possible.

In the simplest case of double symmetry, the three equations and with them the strain quantities ξ_s η_s, and φ themselves, are independent of each other. Accordingly, there are then, three different buckling processes, of which, of course, only the one with the lowest critical compressive stress is of practical significance. The critical compressive stresses $\sigma_{D_1} = \sigma_y$, $\sigma_{D_2} = \sigma_x$, $\sigma_{D_3} = \sigma_s$

$$\xi_s = a_1 \sin \frac{\pi z}{l}, \quad \eta_s = a_2 \sin \frac{\pi z}{l}, \quad \varphi = a_3 \sin \frac{\pi z}{l} \qquad (45a,b,c)$$

at

$$\sigma_y = \frac{\pi^2}{l^2} \frac{E\ J_y}{F}, \quad \sigma_x = \frac{\pi^2}{l^2} \frac{E\ J_x}{F} \qquad (46),(47)$$

and

$$\sigma_s = \frac{G\ J_T + \frac{\pi^2}{l^2} E\ C}{J_p} \qquad (48)$$

Under the critical twisting-failure stress, according to (48) the column thus twists about an axis of rotation, passing through the centroid S. For an I section, for example, it means, with $\frac{E}{G} = 2.6$, if h is the height of the web, and b is the width of the flange:

$$\sigma_s = \frac{\pi^2}{l^2} E \frac{h^2\ b^3 + 0.312\ (h + 2b)\ s^2\ l^2}{4b^3 + 12\ b\ h^2 + 2h^3} \qquad (49)$$

To make twisting failure possible practically, the I section must not only have the proper thin wall, but its flange width must exceed the height of the web; for the condition of smaller σ_s then the Euler stress for buckling about the web axis, reads:

$$\left(\frac{h}{b}\right)^3 + 0.468\ (h + 2b)^2\ s^2\ \frac{l^2}{b^6} \leqq 1 \qquad (50)$$

For sections without axial symmetry but disclosing point symmetry (for example, Z sections with equal legs) (equations 44) also are not coupled because $R_x = R_y = 0$, regardless of whether the centroidal axes x,y are principal axes of inertia or not. And the twisting failure again occurs about an axis of rotation through S under critical compressive stress σ_s, according to equation (48).

In the most general case of asymmetrical open sections, none of the stresses cited in equations (46) to (48) is a critical compressive stress. The critical values σ_{D_1}, σ_{D_2}, σ_{D_3} then follow as roots of a cubic equation.

With simple symmetry their place is taken by a linear and a quadratic equation.

Bleich's article lacks the essential term $\sigma_D(J_x+J_y)\varphi'^2$ in the expression for A_a, though the expression for the strain energy A_i is as to contents in agreement with (36). Closed formulas for the quantities R_y, R_x, C of the type of (37), (38), and (39) are not given; the concept of unit warping itself is not employed. Their investigation is restricted to contoured columns built up from flat plates. They proceed from the curvatures of the individual plates expressed with three strain quantities ξ_s, η_s, φ, and to which the conventional theory of beam flexure is applied. The strains (ε) are made proportional to those curvatures and a linear distribution is postulated along the straight pieces, of which the cross section consists. With n plates it affords for the calculation of the n unknown strains at plate center, n - 1 transition conditions and one equilibrium condition (equation (32)). The calculation of these n unknown strains must be carried through first for each section. And even then the expressions for R_y, R_x, and C (given for some sections by Bleich) follow only after the development of the expression for A_i. The lack of term $\sigma_D(J_x+J_y)\varphi'^2$ in the expression for A_a results in the complete absence of σ_D in Bleich's equation, which corresponds to equation (44c).

3. SPECIAL CASE OF SECTION SYMMETRICAL WITH X AXIS, CHANNEL SECTION AS EXAMPLE

For symmetry with the x axis, it is: $J_{xy} = 0$ and $R_y = 0$. This releases (44a) from (44b) and (44c), which are coupled through $R_x \neq 0$. The solution is again effected with (45) but now a_2 and a_3 are no longer independent of each other. Equation (44a) becomes:

$$E\ J_y\ \xi_s^{IV} + \sigma_D\ F\ \xi_s'' = 0 \tag{51}$$

with which (45a) becomes:

$$\sigma_{D_1} = \sigma_y = \frac{\pi^2}{l^2}\,\frac{E\ J_y}{F} \tag{52}$$

From equations (44b,c), now written:

$$E\ J_x\ \eta_s^{IV} + E\ R_x\ \varphi^{IV} + \sigma_D\ F\ \eta_s'' = 0 \tag{53a}$$

$$E\ R_x\ \eta_s^{IV} + E\ C\ \varphi^{IV} + (\sigma_D\ J_p - G\ J_T)\ \varphi'' = 0 \tag{53b}$$

follow two homogeneous linear equations for the unknown a_2 and a_3

$$\left[\frac{\pi^4}{l^4}\ E\ J_x - \frac{\pi^2}{l^2}\ \sigma_D\ F\right] a_2 + \left[\frac{\pi^4}{l^4}\ E\ R_x\right] a_3 = 0 \tag{54a}$$

$$\left[\frac{\pi^4}{l^4}\ E\ R_x\right] a_2 + \left[\frac{\pi^4}{l^4}\ E\ C + \frac{\pi^2}{l^2}\ G\ J_T - \frac{\pi^2}{l^2}\ \sigma_D\ J_p\right] a_3 = 0 \tag{54b}$$

which have a solution other than zero only when their determinant disappears:

$$\begin{vmatrix} \frac{\pi^2}{l^2}\ E\ J_x - \sigma_D\ F & \frac{\pi^2}{l^2}\ E\ R_x \\ \frac{\pi^2}{l^2}\ E\ R_x & G\ J_T + \frac{\pi^2}{l^2}\ E\ C - \sigma_D\ J_p \end{vmatrix} = 0 \tag{55}$$

This condition affords a quadratic equation for the two critical compressive stresses σ_{D_2} and σ_{D_3}. Abbreviating (47) and (48) with the added abbreviation (dimension of a stress)

$$\sigma_x = \frac{\pi^2}{l^2}\ \frac{E\ R_x}{\sqrt{F\ J_p}} \tag{56}$$

equation (55) can be written as:

$$\begin{vmatrix} \sigma_x - \sigma_D & \rho_x \\ \rho_x & \sigma_s - \sigma_D \end{vmatrix} = \sigma_D^2 - \sigma_D\ (\sigma_s + \sigma_x) + \sigma_s\ \sigma_x - \rho_x^2 = 0 \tag{57}$$

that is:

$$\sigma_D = \frac{\sigma_s + \sigma_x}{2} \mp \sqrt{\left(\frac{\sigma_s - \sigma_x}{2}\right)^2 + \rho_x^2} \tag{58}$$

This relation can be represented by Mohr's circle (fig. 3) and thus enable a simple graphical solution of σ_D from σ_s, σ_x, and ρ_x. The smaller root is seen to be consistently smaller, the greater root consistently greater than σ_x and σ_s. The strains correlated to σ_{D_2} and σ_{D_3} are given with equations (45b,c), from which follows:

$$\eta_s = \frac{a_2}{a_3}\varphi = \tilde{x}_s \varphi = - x_D \varphi \qquad (59)$$

The cross sections accordingly twist about the centers of rotation D_2 and D_3 lying on the axis of symmetry and whose coordinates x_{D_2} and x_{D_3}, after insertion of σ_{D_2} and σ_{D_3} in one of (54), give:

$$x_D = - \frac{a_2}{a_3} = \frac{\frac{\pi^2}{l^2} E R_x}{\frac{\pi^2}{l^2} E J_x - \sigma_D F} = \sqrt{\frac{J_p}{F}} \frac{\rho_x}{\sigma_x - \sigma_D} \qquad (60)$$

or else

$$x_D = - \frac{a_2}{a_3} = \frac{G J_T + \frac{\pi^2}{l^2} E C - \sigma_D J_p}{\frac{\pi^2}{l^2} E R_x} = \sqrt{\frac{J_p}{F}} \frac{\sigma_s - \sigma_D}{\rho_x} \qquad (61)$$

For $\rho_x \neq 0$ it is $\sigma_D \neq \sigma_x$ and $\sigma_D \neq \sigma_s$: the value for x_D is, in consequence, always finite. In other words, we obtain the important result that on a symmetrical open section a buckling in direction (y) perpendicular to the axis of symmetry (x) is always accompanied by column twist. For very thick-walled sections, of course, σ_s is very great on account of the great twisting strength so that $\sigma_{D_2} \approx \sigma_x$ and $\sigma_{D_3} \approx \sigma_s$. This is recognized from Mohr's circle or else from the approximate formulas valid for great ratio σ_s/σ_x:

$$\sigma_{D_2} \approx \sigma_x - \frac{\rho_x^2}{\sigma_s} \qquad (62)$$

$$\sigma_{D_3} \approx \sigma_s + \frac{\rho_x^2}{\sigma_s} \qquad (63)$$

following from (57) or (58). In this case, for $\sigma_{D_2} \approx \sigma_x$ the center of rotation D_2 is extremely remote from the centroid, while for very high twisting-failure stress $\sigma_{D_3} \approx \sigma_s$, the center of rotation D_3 is almost coincident with the centroid. On sufficiently thick-walled, symmetrical open sections, the buckling in direction perpendicular to the axis of symmetry is therefore practically free from twist.

Bleich, lacking the term $-\sigma_D J_p$ in the determinant (55), reached instead of (57), the linear equation:

$$(\sigma_x - \sigma_{Bleich})\,\sigma_s - \rho_x^2 = 0 \tag{64}$$

with the single solution[7)]

$$\sigma_{Bleich} = \sigma_x - \frac{\rho_x^2}{\sigma_s} \tag{65}$$

This formula which, with observance of

$$\frac{\rho_x}{\sigma_x} = \frac{R_x\sqrt{F}}{J_x\sqrt{J_p}} = \frac{x_M}{i_p} \tag{66}$$

(equations 47, 56, and 15) may also be written as

$$\sigma_{Bleich} = \sigma_x\left(1 - \frac{x_M^2}{i_p^2}\,\frac{\sigma_x}{\sigma_s}\right) \tag{67}$$

agrees with the approximate formula (62), valid for very small ratio σ_x/σ_s, and supplies for that reason useful values for the conventional sections and lengths used in structural engineering. But for the thin-walled sections customary in airplane design, the discrepancies between σ_{Bleich} and σ_D may become quite considerable. From the theoretical point of view, it is even more essential that, according to Bleich's formula for A_a, one of the critical compressive stresses always be lost. According to that theory, for instance, twisting failures of point symmetrical sections should be impossible.

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7) Bleich's equation of the critical load, written with our notation, reads as follows:

$$P_{Bleich} = P_E\left(1 - \frac{R_x^2}{J_x\left(c + \dfrac{G\,J_T\,J_x}{P_E}\right)}\right),\quad P_E = \frac{\pi^2\,E\,J_x}{l^2}$$

The boundary conditions for the case corresponding to (58) are, as subsequently established from (45) for both cases $(z = 0, \quad z = l)$: $\eta_s = 0$, $\eta_s'' = 0$ and $\varphi = 0$, $\varphi'' = 0$. It means, according to (31), that $\tilde{\sigma} = 0$; that is, complete axial nonconstraint obtains. For it, $\tilde{W} = -\varphi' \tilde{w}$ with $\varphi' \neq 0$, according to equation (30). The exact realization of these boundary conditions, even on the simplest cross-sectional forms, would require special supports with several knife edges, in order to enable the individual parts of the support to follow the cross-sectional warping.

With complete axial constraint (say, by welding to rigid support plates, etc.), the following formulas replacing (45) are appropriate:

$$\eta_s = a_2 \left(1 - \cos \frac{2\pi z}{l}\right), \quad \varphi = a_3 \left(1 - \cos \frac{2\pi z}{l}\right) \tag{68}$$

At both boundaries we have $\varphi = 0$ and $\varphi' = 0$; that is, $\tilde{W} = 0$, according to equation (30), and in addition, $\eta_s = 0$ and $\eta_s' = 0$. As $\varphi'' \neq 0$, $\tilde{\sigma}$ likewise is $\tilde{\sigma} \neq 0$, according to equation (31). Everything else follows in the same manner as before; in equations (54) to (58) $l/2$ replaces l.

For other boundary or support conditions, it is recommended to obtain by tests a substitute length which replaces l in (58). The substitute length lies always between $l/2$ and l.

For illustration, we repeat the formulas for a channel section. The solution of the unit warping, of warping moment E_x, and warping strength C, is carried out in section 8. With a as web height, and b as flange length, it is:

$$\sigma_x = \frac{\pi^2 E}{l^2} \frac{a^2(a+6b)}{12(a+2b)}, \qquad \sigma_y = \frac{\pi^2 E}{l^2} \frac{b^3(2a+b)}{3(a+2b)^2} \qquad (69), (70)$$

$$\sigma_s = \frac{\pi^2 E}{l^2} \frac{a^2 b^3 (2a^2+15ab+26b^2) + \frac{4}{\pi} \frac{G}{E}(a+2b)^3 \, s^2 \, l^2}{(a+2b)\left\{a^2(a+6b)\,(a+2b) + 4b^3(2a+b)\right\}} \tag{71}$$

$$\rho_x = \frac{\pi^2 E}{l^2} \frac{2a^2 \, b^2 \, (a+3b)}{(a+2b)\sqrt{3\left\{a^2(a+6b)\,(a+2b) + 4b^3(2a+b)\right\}}} \tag{72}$$

In the special case of $a = b$, illustrated in figure 4, we obtain (with $\frac{E}{G} = 2.6$):

$$\sigma_x = \frac{7}{36} \frac{\pi^2 E}{(l/b)^2} = 0.1944 \frac{\pi^2 E}{(l/b)^2} \tag{73}$$

$$\sigma_y = \frac{1}{9} \frac{\pi^2 E}{(l/b)^2} = 0.1111 \frac{\pi E}{(l/b)^2} \tag{74}$$

$$\sigma_s = \left[0.4345 + 0.04251 \left(\frac{s}{b}\right)^2 \left(\frac{l}{b}\right)^2\right] \frac{\pi^2 E}{(l/b)^2} \tag{75}$$

$$\rho_x = - 0.2680 \frac{\pi^2 E}{(l/b)^2} \tag{76}$$

Then σ_D follows, according to equation (58) or with the aid of Mohr's circle. We have included in figure 4 the curves for Wagner's formula[8]:

$$\sigma_M = \frac{G\ J_T + \frac{\pi^2}{l^2} E \cdot C^*}{J_p^*} \tag{77}$$

as well as Bleich's formula (65) or (67) for comparison. For $a = b$, is obtained:

$$\sigma_M = \left[0.02239 + 0.01466 \left(\frac{s}{b}\right)^2 \left(\frac{l}{b}\right)^2\right] \frac{\pi E}{(l/b)^2} \tag{78}$$

and

$$\sigma_{Bleich} = 0.1944 \left[1 - \frac{0.3693}{0.4345 + 0.04251 \left(\frac{s}{b}\right)^2 \left(\frac{l}{b}\right)^2}\right] \frac{\pi^2 E}{(l/b)^2} \tag{79}$$

The discrepancies between σ_M and σ_D as explained in section 6, increase with wall thickness and length, while the differences between σ_{Bleich} and σ_D are great-

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[8] C^* is identical with Wagner's C_{bdu}. He then added (disregarded here) the term C_{bdn} connected with the transverse warping, but which serves no useful purpose unless it precisely pertains to very short columns of unflanged angle section, T section, or cruciform section.

est for small length and wall thickness. The curve of the twisting failure stresses σ_D cannot approach the Euler curve (σ_x) for buckling perpendicular to the axis of symmetry, except asymptotically, according to equation (62), although it can intersect the Euler curve (σ_y) for buckling in direction of the axis of symmetry. This is the case for so much shorter lengths as the section wall is thicker.

4. THE GENERAL CASE

If the cross section reveals neither axial nor point symmetry, the warping moments ξ_s, η_s, φ do not disappear and equations (44a,b,c) are, as a result, coupled. In equation (45)

$$\xi_s = a_1 \sin \frac{\pi z}{l}, \quad \eta_s = a_2 \sin \frac{\pi z}{l}, \quad \varphi = a_3 \sin \frac{\pi z}{l}$$

therefore the unknown coefficients a_1, a_2, a_3, are mutually dependent and result in three linear, homogeneous equations, whose determinant must disappear for the critical case. This relation supplies a cubic equation for the three potential twisting-failure stresses σ_{D_1}, σ_{D_2}, σ_{D_3}, of which, of course, only the smallest retains our interest. The insertion of the σ_D values gives three different ratios a_1/a_3 and a_2/a_3 each, which define the position of the three centers of rotation D_1, D_2, and D_3, as is seen from the following equations:

$$\left.\begin{aligned} \xi_s &= \frac{a_1}{a_3}\,\varphi = -\,\tilde{y}_s\,\varphi = +\,y_D\,\varphi, \quad \text{so that} \quad y_D = \frac{a_1}{a_3} \\ \eta_s &= \frac{a_2}{a_3}\,\varphi = +\,\tilde{x}_s\,\varphi = -\,x_D\,\varphi, \quad \text{so that} \quad x_D = -\,\frac{a_2}{a_3} \end{aligned}\right\} \quad (80)$$

With the abbreviations (46), (47), (48), and (56), as well as

$$\rho_y = \frac{\pi^2}{l^2}\,\frac{E\,R_y}{\sqrt{F}\,J_p} \qquad (81)$$

$$\beta_{xy} = \frac{\pi^2}{l^2}\,\frac{E\,J_{xy}}{F} \qquad (82)$$

the disappearing determinant

$$\begin{vmatrix} \frac{\pi^2}{l^2} E J_y - \sigma_D F & \frac{\pi^2}{l^2} E J_{xy} & \frac{\pi^2}{l^2} E R_y \\ \frac{\pi^2}{l^2} E J_{xy} & \frac{\pi^2}{l^2} E J_x - \sigma_D F & \frac{\pi^2}{l^2} E R_x \\ \frac{\pi^2}{l^2} E R_y & \frac{\pi^2}{l^2} E R_x & \frac{\pi^2}{l^2} E C + G J_T - \sigma_D J_p \end{vmatrix} = 0$$

may be written in the form:

$$\begin{vmatrix} \sigma_y - \sigma_D & \rho_{xy} & \rho_y \\ \rho_{xy} & \sigma_x - \sigma_D & \rho_x \\ \rho_y & \rho_x & \sigma_s - \sigma_D \end{vmatrix} = 0 \qquad (83)$$

Equation (83) is the well-known "secular equation" which reads:

$$\sigma_D^3 - \sigma_D^2(\sigma_s+\sigma_x+\sigma_y) + \sigma_D(\sigma_s\sigma_x + \sigma_s\sigma_y + \sigma_x\sigma_y -$$

$$- \rho_{xy}^2 - \rho_x^2 - \rho_y^2) - (\sigma_s\sigma_x\sigma_y + 2\rho_{xy}\rho_x\rho_y -$$

$$- \rho_{xy}^2 \sigma_s - \rho_y^2 \sigma_x - \rho_x^2 \sigma_y) = 0 \qquad (84)$$

All roots are real (symmetrical determinant!) That they are always positive as well, is seen from the following. The integrands of A_i and A_a in the strain quantities ξ_s, η_s, φ and their derivatives, respectively, being quadratic and homogeneous, A_i and A_a become through (45) homogeneous quadratic functions of the coefficients a_1, a_2, and a_3. The result is:

$$A_i = l \frac{\pi^2}{l^2} \left[\frac{\pi^2}{l^2} E J_y a_1^2 + 2 \frac{\pi^2}{l^2} E J_{xy} a_1 a_2 + \frac{\pi^2}{l^2} E J_x a_2^2 + \right.$$

$$\left. + 2 \frac{\pi^2}{l^2} E R_y a_1 a_3 + 2 \frac{\pi^2}{l^2} E R_x a_2 a_3 + \left(G J_T + \frac{\pi^2}{l^2} EC\right) a_3^2 \right]$$

$$= \Phi (a_1, a_2, a_3) \qquad (85)$$

and

$$A_a = l \frac{\pi^2}{l^2} \sigma_D [F(a_1^2 + a_2^2) + J_p a_3^2] =$$

$$= \sigma_D \psi(a_1, a_2, a_3) \tag{86}$$

Since the difference $A_i - A_a$ and with it $\Phi - \sigma_D \psi$ shall be a minimum, the three equations

$$\frac{\partial(\Phi - \sigma_D \psi)}{\partial a_i} = 0, \quad i = 1, 2, \text{ and } 3 \tag{87}$$

must be complied with. Obviously, these are identical with those obtained from (44) through (45), whose disappearing determinant leads to equation (84) and so, to the three values σ_{D_1}, σ_{D_2}, σ_{D_3} and to the three ratios $a_1 : a_2 : a_3$. To $\Phi - \sigma_D \psi$ as homogeneous quadratic function of the a_i, the Euler formula:

$$\frac{\partial(\Phi - \sigma_D \psi)}{\partial a_1} a_1 + \frac{\partial(\Phi - \sigma_D \psi)}{\partial a_2} a_2$$

$$+ \frac{\partial(\Phi - \sigma_D \psi)}{\partial a_3} a_3 = 2(\Phi - \sigma_D \psi) \tag{88}$$

is applicable; that is, the difference, because of the three equations (87) is

$$\Phi - \sigma_D \psi = 0 \tag{89}$$

which in the present case proves the correctness of the double statement $A_i - A_a = \min = 0$. Such a proof can be quite generally adduced when introducing

$$\xi_s = a_1 f_1(z), \quad \eta_s = a_2 f_2(z), \quad \varphi = a_3 f_3(z) \tag{90}$$

in place of (45), because Φ and ψ are again always homogeneous quadratic functions of a_1, a_2, and a_3. The solution of equation (89) with respect to σ_D gives:

$$\sigma_D = \frac{\Phi(a_1, a_2, a_3)}{\psi(a_1, a_2, a_3)} = \frac{\bar{\Phi}\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}\right)}{\bar{\psi}\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}\right)} \tag{91}$$

where, of course, the values $\frac{a_1}{a_3}$ and $\frac{a_2}{a_3}$ should be inserted (but which follow from (87), not before). Having thus defined ψ and Φ as positive, as is readily seen for ψ from (86) and for ψ from its physical significance, it follows from equation (91) that the three roots σ_{D_1}, σ_{D_2}, and σ_{D_3} are all positive (i.e., compressive stresses).

The premise $A_1 - A_a = \min = 0$ can, moreover, be replaced by the other demand:

$$\sigma_D = \min\left[\frac{\Phi(\xi_s, \eta_s, \varphi)}{\psi(\xi_s, \eta_s, \varphi)}\right] \tag{92}$$

wherein ξ_s, η_s, φ would be the solution functions (unknown as yet) of the problem. The differential equations for these functions and also of σ_D itself, are obtained from equation (92). The variation carried through, gives:

$$\psi\,\delta\,\Phi - \Phi\,\delta\,\psi = 0 \tag{93}$$

$$\delta\,\Phi - \frac{\Phi}{\psi}\,\delta\,\psi = \delta(\Phi - \sigma_D\,\psi) = \delta(A_1 - A_a) = 0 \tag{94}$$

which proves the previous argument.

The practical solution of (84) is best effected with the graph for the cubic equation given as figure 5:

$$\omega^3 - 3\omega^2 + p\,\omega - q = 0 \tag{95}$$

Hereby:

$$p = \frac{\sigma_s\,\sigma_x + \sigma_s\,\sigma_y + \sigma_x\,\sigma_y - \rho_{xy}^2 - \rho_x^2 - \rho_y^2}{\sigma_m^2} \tag{96}$$

$$q = \frac{\sigma_s\,\sigma_x\,\sigma_y + 2\rho_{xy}\,\rho_x\,\rho_y - \rho_{xy}^2\,\sigma_s - \rho_y^2\,\sigma_x - \rho_x^2\,\sigma_y}{\sigma_m^3} \tag{97}$$

$$\sigma_m = \frac{\sigma_s + \sigma_x + \sigma_y}{3} \tag{98}$$

First determine: σ_s, σ_x, σ_y, then σ_m, p, q, and read the particular minimum ω value from the chart. The twisting-failure stress then follows as

$$\sigma_D = \omega \, \sigma_m \tag{99}$$

For very thick-walled sections, the conditions resemble those of the special case discussed previously. One of the three roots, say, σ_{D_3}, is approximately equal to the very high stress σ_s, while the other two roots follow from equation (84) divided by σ_s. The buckling here is practically free from twist. With x and y as principal axes, for example, $(\rho_{xy} = 0)$, it is $\sigma_{D_1} \approx \sigma_x$ and $\sigma_{D_2} \approx \sigma_y$.

5. REFERENCE TO ANY REFERENCE POINT O; TWISTING FAILURE FOR FORCED AXIS OF ROTATION A

The cross-sectional shape being preserved, the description of the strains can be made with three (on z dependent) sets of data. This time, however, the chosen strain quantities include, aside from the angle of twist, the displacements ξ_o, η_o of an arbitrary reference point O. O is to have the coordinates x_o, y_o and from the origin of a coordinate system $\bar{x}$, $\bar{y}$ parallel to x, y. The unit warping $\tilde{w}$ referred to the real center of rotation D is expressed, according to figure 2 (cf. equation 9), as:

$$\tilde{w} = \bar{w} + \tilde{x}_o \, y - \tilde{y}_o \, x + K \tag{100}$$

Hereby the unit warping $\bar{w}$ referred to O is given through

$$\bar{w} = \bar{w}_o + \int_o^u \bar{r}_t \, du \tag{101}$$

Further, it is

$$\xi_o = - \tilde{y}_o \, \varphi, \qquad \eta_o = + \tilde{x}_o \, \varphi \tag{102}$$

that is:

$$\bar{\epsilon} = - \xi_o'' \, x - \eta_o'' \, y - \varphi'' \left(\bar{w} - \frac{1}{F} \int_F \bar{w} \, dF \right) \tag{103}$$

The result is:

$$A_i = \frac{1}{2} \int_{z=0}^{z=l} \left\{ E\ J_y\ \xi_o''^2 + 2E\ J_{xy}\ \xi_o''\ \eta_o'' + E\ J_x\ \eta_o''^2 + \right.$$

$$+ 2E\ \overline{R}_y\ \xi_o''\ \varphi'' + 2E\ \overline{R}_x\ \eta_o''\ \varphi'' +$$

$$\left. + E\ \overline{C}\ \varphi''^2 + G\ J_{\underline{T}}\ \varphi'^2 \right\} dz \tag{104}$$

with

$$\overline{R}_y = \int_F x\ \overline{w}\ dF \tag{105}$$

$$\overline{R}_x = \int_F y\ \overline{w}\ dF \tag{106}$$

$$\overline{C} = \int_F \overline{w}^2\ dF - \frac{1}{F}\left(\int_F w\ dF\right)^2 \tag{107}$$

Moreover:

$$\xi = \xi_o - \overline{y}\ \varphi, \qquad \eta = \eta_o + \overline{x}\ \varphi \tag{108}$$

and

$$\overline{y}_s = -\ y_o, \qquad \overline{x}_s = -\ x_o \tag{109}$$

$$A_a = \sigma_D\ \frac{1}{2} \int_{z=0}^{z=l} \left\{ F\ (\xi_o'^2 + \eta_o'^2) + \overline{J}_p\ \varphi'^2 + \right.$$

$$\left. + 2F\ y_o\ \xi_o'\ \varphi' - 2F\ x_o\ \eta_o'\ \varphi' \right\} dz \tag{110}$$

Herewith follow from $\delta(A_i - A_a) = 0$, the three differential equations:

$$EJ_y\ \xi_o^{IV} + EJ_{xy}\ \eta_o^{IV} + E\overline{R}_y\ \varphi^{IV} + \sigma_D\ F\ \xi_o'' + \sigma_D\ F\ y_o\ \varphi'' = 0 \tag{111a}$$

$$E J_{xy}\ \xi_o^{IV} + E J_x\ \eta_o^{IV} + E\bar{R}_x\ \varphi^{IV} + \sigma_D\ F\ \eta_o'' - \sigma_D\ F\ x_o\ \varphi'' = 0 \qquad (111b)$$

$$E\bar{R}_y\ \xi_o^{IV} + E\bar{R}_x\ \eta_o^{IV} + E\bar{C}\ \varphi^{IV} + \sigma_D\ F\ y_o\ \xi_o'' - \sigma_D\ F\ x_o\ \eta_o'' +$$

$$+ (\sigma_D\ \bar{J}_p - G\ J_T)\ \varphi'' = 0 \qquad (111c)$$

These equations (111) are more complicated than equation (44), but they can be considerably simplified when introducing the actual center of rotation D as arbitrary reference point O, because

$$\xi_D \equiv 0, \quad \eta_D \equiv 0 \qquad (112)$$

They then become:

$$E\ \tilde{R}_y\ \varphi^{IV} + \sigma_D\ F\ y_D\ \varphi'' = 0 \qquad (113a)$$

$$E\ \tilde{R}_x\ \varphi^{IV} - \sigma_D\ F\ x_D\ \varphi'' = 0 \qquad (113b)$$

$$E\ \tilde{C}\ \varphi^{IV} + (\sigma_D\ \tilde{J}_p - G\ J_T)\ \varphi'' = 0 \qquad (113c)$$

whereby the warping moments:

$$\tilde{R}_y = \int_F x\ \tilde{w}\ dF \qquad (114)$$

and

$$\tilde{R}_x = \int_F y\ \tilde{w}\ dF \qquad (115)$$

as well as the warping strength $\tilde{C}$ depend on the (yet unknown) position of the center of rotation D. Equation (113c) is in accord with equation (26), obtained from the condition for moment equilibrium about the axis of rotation of a column element. The equilibrium condition $\Sigma Z = 0$ for the forces in column direction is already met by equation (32)

$$\int_F \tilde{w}\ dF = 0$$

thus eliminating the second term in equation (107) and leaving for $\tilde{C}$ the simpler equation (25):

$$\tilde{C} = \int_F \tilde{w}^2 \, dF$$

Equations (113a,b) are the previously cited conditions for equilibrium of the moments about the y and x axes in connection with the equilibrium conditions $\Sigma X = 0$ and $\Sigma Y = 0$ for any column element. This is readily seen from the expressions for the bending moments and transverse forces. With equation (31) the bending moments are:

$$B_y = \int_F x \, \tilde{\sigma} \, dF = - E \, \tilde{R}_y \, \varphi'' \tag{116}$$

and

$$B_x = \int_F y \, \tilde{\sigma} \, dF = - E \, \tilde{R}_x \, \varphi'' \tag{117}$$

and the transverse forces:

$$Q_x = - \int_{u=u_0}^{u=u_R} \tilde{\tau} \, s \, du \sin \alpha = + \int_F \tilde{\tau} \, s \, dx = - E \, \tilde{R}_y \, \varphi''' \tag{118}$$

and

$$Q_y = + \int_{u=u_0}^{u=u_R} \tilde{\tau} \, s \, du \cos \alpha = + \int_F \tilde{\tau} \, s \, dy = - E \, \tilde{R}_x \, \varphi''' \tag{119}$$

when applying the relation:

$$\tau \, s = + E \, \varphi''' \int_{u=u_0}^{u} \tilde{w} \, dF \tag{120}$$

following from equation (21) and effecting a partial integration.

Writing the expression (9) or (9') into (114), (115), and (25), gives:

$$\tilde{R}_y = R_y + \tilde{x}_s \, J_{xy} - \tilde{y}_s \, J_y = R_y - x_D \, J_{xy} + y_D \, J_y \tag{121}$$

$$\tilde{R}_x = R_x + \tilde{x}_s \, J_x - \tilde{y}_s \, J_{xy} = R_x - x_D \, J_x + y_D \, J_{xy} \tag{122}$$

and
$$\tilde{C} = C + x_D^2\,J_x - 2x_D\,y_D\,J_{xy} + y_D^2\,J_y$$
$$- 2x_D\,R_x + 2y_D\,R_y \tag{123}$$

Then oquation (113) can bo solved with

$$\varphi = \tilde{a}\,\sin\frac{\pi z}{l} \tag{124}$$

yielding at first:

$$\sigma_D = \frac{\pi^2}{l^2}\,\frac{E\tilde{R}_y}{F y_D}, \qquad \sigma_D = -\frac{\pi^2}{l^2}\,\frac{E\tilde{R}_x}{F x_D} \tag{125), (126}$$

and

$$\sigma_D = \frac{G\,J_T + \frac{\pi^2}{l^2}\,E\,\tilde{C}}{\tilde{J}_p} \tag{127}$$

By oliminating σ_D from each of two equations, the unknown distances x_D and y_D of the center of rotation can be determined, after which insertion in one of the three (125) to (127) gives σ_D. Compared with (95) to (99), this method is extremely tiresome, since the equations for x_D and y_D are coupled and of the third degree.

Admittedly, considerable simplification obtains if symmetry prevails. With $R_y = 0$ and $y_D = 0$, for example, equations (126) and (127) bocome:

$$\sigma_D = -\frac{\pi^2}{l^2}\,\frac{E\,R_x \quad x_D\,J_x)}{F\,x_D} \tag{128}$$

and

$$\sigma_D = \frac{G\,J_T + \frac{\pi^2}{l^2}\,E(C - 2x_D\,R_x + x_D^2\,J_x)}{J_p + F\,x_D^2} \tag{129}$$

From this follows as distance x_D of the axis of rotation, from the center of gravity, the quadratic equation

$$x_D^2 - x_D\,\frac{C + \frac{1}{\pi^2}\,\frac{G}{E}\,J_T\,l^2 - \frac{J_p}{F}\,J_x}{R_x} - \frac{J_p}{} = 0 \tag{130}$$

Owing to the term $\left(\frac{1}{\pi^2}\right)\left(\frac{G}{E}\right)$ $J_T\, l^2$, x_D is not only dependent on the cross-sectional parameters given through the section center line, but also on the wall thickness and the column length (parameter $\left(\frac{s}{b}\right)^2\left(\frac{l}{b}\right)^2$, where b denotes any cross-sectional dimension.[9])

Effecting any axis of rotation A intersecting the axis of symmetry - by hingelike guidances along the whole bar (fig. 6a), for instance - affords the following equation:

$$\sigma_A = \frac{G\ J_T + \frac{\pi^2}{l^2}\ E(C - 2x_A\ R_x + x_A^2\ J_x)}{J_p + F\ x_A^2} \tag{131}$$

which springs from equation (129), if x_D is replaced by x_A, the distance of the axis of rotation A from the column axis passing through S. For, as is readily seen, the reaction forces appearing on the guides have no effect on the equilibrium of the moments about the axis of rotation A, and equations (26), (113d), and (129) retain their validity, if A is used instead of D. But now equations (113a,b) are no longer applicable, because the reactions modify the transverse forces in the individual cross sections. Though the energy expressions A_i and A_a (equations (104) and (110)) remain the same, only φ may be varied in $\delta(A_i - A_a) = 0$, as φ now constitutes the sole independent quantity.

Figure 7 illustrates the relation of critical compressive stress σ_A to distance x_A for a certain channel section. The extreme points of the curve $\sigma_A = f(x_A)$ are at the same time the zero points of equation (130), as

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[9] While reading the proofs of this article, a report touching on the same subject by E. E. Lundquist, entitled: "On the Strength of Columns that Fail by Twisting," appeared in The Journal of the Aeronautical Sciences, vol. IV, no. 6, April 1937. Referring to H. Wagner (reference 4), Lundquist proceeds from equation (127) and postulates for the axis of rotation such a position as will cause σ_D to be a minimum. This method leads to the solution of an equation system - (125) to (127) - (or of (128) and (129) for symmetry).

can be proved from a differentiation. The two roots of equation (57) therefore agree with the extreme values of equation (131). This is directly connected with the previous formulation of the energy method in equation (92), because the numerator or denominator, respectively, of equation (131), is proportional to the respective internal or external work in establishing the sine formula (124).

For $x_A = \infty$, σ_A is equal to the Euler stress σ_E. This case can be enforced, for example, with guidances of the type of figure 6b. The result of the force couples due to the guides is that the transverse forces applied at the cross sections always assume the position demanded by the equilibrium. On the channel section of figure 6b, for example, the transverse force Q_y must pass through S on account of the central compressive force, which is impossible without guides.

The cases cited in figures 6 and 7 are of significance insofar as columns of open section are frequently used as stiffeners of the skin of stressed-skin structures. The axis of rotation is in this case subject to considerable displacement toward the sheet which in addition furnishes an elastic support against twisting, resulting in a marked rise of critical compressive stress compared to the values computed here. A mathematical analysis of the two effects on combined action of metal skin and contoured columns is, of course, quite difficult.

6. RELATION OF TWISTING FAILURE STRESS σ_D TO WAGNER'S CRITICAL STRESS σ_W

With center of shear M chosen as reference point in the sense of the preceding chapter, we obtain, in place of equation (111), three corresponding differential equations for the displacements of the shear center ξ_M, η_M, and the angle of twist φ. With the unit warping w^* as defined in equation (11) and referred to the shear center, C^* now replaces $\overline{C}$, that is:

$$C^* = \int_F w^{*2}\, dF - \frac{1}{F}\left(\int_F w^*\, dF\right)^2 \qquad (132)$$

Owing to equation (10), the moments R_y^* and R_x^* replacing $\bar{R}_y$ and $\bar{R}_x$ disappear, which affords a certain simplification over equation (111):

$$R_y^* \equiv \int_F x\, w^*\, dF = 0, \quad R_x^* \equiv \int y\, w^*\, dF = 0 \qquad (133)$$

Only the symmetrical case is treated hereafter. For symmetry with the x axis, we obtain with $J_{xy} = 0$, $y_M = 0$, the following equations from equations (111b,c):[10)]

$$E\, J_x\, \eta_M^{IV} + \sigma_D\, F\, \eta_M'' - \sigma_D\, F\, x_M\, \varphi'' = 0 \qquad (134a)$$

$$E\, C^*\, \varphi^{IV} + (\sigma_D\, J_p^* - G\, J_T)\varphi'' - \sigma_D\, F\, x_M\, \eta_M'' = 0 \qquad (134b)$$

Using the abbreviation σ_M for Wagner's critical stress, according to equation (77), in conjunction with equation (45) for the critical compressive stress σ_D, we have:

$$\begin{vmatrix} \sigma_x - \sigma_D & \sigma_D \dfrac{x_M}{i_p^*} \\ \sigma_D \dfrac{x_M}{i_p^*} & \sigma_M - \sigma_D \end{vmatrix} = \sigma_D^2 \left(1 - \frac{x_M^2}{i_p^{*2}}\right) - \sigma_D(\sigma_M + \sigma_x) + \sigma_M \sigma_x = 0 \qquad (135)$$

that is:

$$\sigma_D = \frac{1}{2\left(1 - \dfrac{x_M^2}{i_p^{*2}}\right)} \left\{\sigma_x + \sigma_M \mp \sqrt{(\sigma_x - \sigma_M)^2 + 4\sigma_x\, \sigma_M \frac{x_M^2}{i_p^{*2}}}\right\} \qquad (136)$$

or else:

$$\frac{1}{\sigma_D} = \frac{1}{2}\left\{\frac{1}{\sigma_M} + \frac{1}{\sigma_x} \pm \sqrt{\left(\frac{1}{\sigma_M} - \frac{1}{\sigma_x}\right)^2 + \frac{4}{\sigma_M\, \sigma_x} \frac{x_M^2}{i_p^{*2}}}\right\} \qquad (136')$$

If $x \neq 0$, neither σ_M nor σ_x are critical stresses, as seen from equations (135) to (136'). The relations:

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10) Cf. equations (53a,b).

$$\frac{\sigma_M}{\sigma_D} = \frac{1}{2}\left\{1 + \frac{\sigma_M}{\sigma_x} + \sqrt{\left(1 - \frac{\sigma_M}{\sigma_x}\right)^2 + 4\,\frac{\sigma_M}{\sigma_x}\,\frac{x_M{}^2}{i_p{}^{*2}}}\right\} \tag{137}$$

and

$$\frac{\sigma_x}{\sigma_D} = \frac{1}{2}\left\{1 + \frac{\sigma_x}{\sigma_M} + \sqrt{\left(1 - \frac{\sigma_x}{\sigma_M}\right)^2 + 4\,\frac{\sigma_x}{\sigma_M}\,\frac{x_M{}^2}{i_p{}^{*2}}}\right\} \tag{138}$$

following from equation (136') are illustrated in figure 8. It immediately affords the percentage of error incurred when the critical compressive stress σ_D is computed by Wagner's formula (77) or according to Euler's (47). The error increases with the ratio, respectively, of σ_M/σ_x or σ_x/σ_M and with the ratio $\sigma_M/i_p{}^*$, dependent on the section form as, for example, is shown for channel sections against aspect ratio a/b in figure 9. The discrepancies between σ_D and σ_M, and between σ_D and σ_x, respectively, are least when the ratio

$$\frac{\sigma_M}{\sigma_x} = \frac{C^* + \frac{1}{\pi^2}\,\frac{G}{E}\,J_T\,l^2}{J_p\,i_x{}^2} \tag{139}$$

is a minimum or maximum, respectively; i.e., in the case of vanishing wall thickness or column length, respectively, for very great wall thicknesses or column lengths. In these cases the approximate formulas:

$$\frac{\sigma_M}{\sigma_D} \approx 1 + \frac{\sigma_M}{\sigma_x}\,\frac{x_M{}^2}{i_p{}^{*2}} \tag{140}$$

or

$$\frac{\sigma_x}{\sigma_D} \approx 1 + \frac{\sigma_x}{\sigma_M}\,\frac{x_M{}^2}{i_p{}^{*2}} \tag{141}$$

are applicable. The latter is in agreement with the approximate formula (62) and with equation (67) for σ_{Bloich}. For the case $\sigma_M = \sigma_x$,

$$\left(\frac{\sigma_M}{\sigma_D}\right)_{\sigma_M/\sigma_x=1} = \left(\frac{\sigma_x}{\sigma_D}\right)_{\sigma_x/\sigma_M=1} = 1 + \frac{x_M}{i_p{}^*} \tag{142}$$

is exactly valid.

If limited to values below 1 for σ_M/σ_x (shorter column lengths and wall thicknesses), respectively, and for σ_x/σ_M (greater column lengths and wall thicknesses), the error for σ_D conformable to (77) and (47), respectively, cannot exceed 100 percent, according to equation (142). The maximum error in the most unfavorable case is 97 percent for channel sections, according to figure 9, and 82 percent for the case $\left(\frac{a}{b} = 1\right)$, illustrated in figure 4. In reality the maximum error here does not exceed 44 percent (fig. 4), because the Euler stress σ_y for buckling in symmetry direction becomes less than σ_M for fairly short lengths, so that starting with such lengths the critical stress would then no longer be computed by Wagner's formula (77) but by the Euler formula (46). Such limitation of the maximum possible error exists, for example, on all channel sections with $a/b > 0.73$ $(\sigma_y < \sigma_x)$. Figure 8 shows the maximum values of σ_M/σ_D for channel sections with $a/b > 0.73$ as a dashed curve. For the rest, the use of figure 8 in conjunction with Wagner's formula (77), is of real advantage only if tables for C^*, J^*, and x_M are available; otherwise, equation (58) gives quicker results.

It also will be noted that the Wagner stress σ_M is directly obtainable from the energy method through the arbitrary (Ritz's) formula:

$$\xi_M = 0, \quad \eta_M = 0, \quad \varphi = a_3 \sin \frac{\pi z}{l} \tag{143}$$

which, written in (104) and (110) for A_i and A_a - center of shear M replacing reference point 0 - gives:

$$A_i - A_a = \Phi(0,\ 0,\ a_3) - \sigma_D\, \Psi(0,\ 0,\ a_3) > 0 \tag{144}$$

that is

$$\sigma_D < \frac{\Phi(0,\ 0,\ a_3)}{\Psi(0,\ 0,\ a_3)} \tag{145}$$

Then the equal sign, replacing the unequal sign in (144) and (145), gives the Wagner stress σ_M. The reason that, in spite of the doubtfulness of the method of making arbitrary assumptions for several variables, the errors are, on the whole, not excessive, is probably due to the fact that in (145) - accidentally, at least - the numerator Φ was made a minimum through (143), because it consists of only two quotas:

$$\frac{1}{2}\int_0^l E\ C^*\ \varphi''^2\ dz \quad \text{and} \quad \frac{1}{2}\int_0^l G\ J_T\ \varphi'^2\ dz$$

and from C^* it can readily be shown that under all possible values it has the smallest.[11] For angle and T sections, for example, $C^* = 0$.

Reverting to figure 7 and equation (131), applicable in the case of symmetry with the x axis: While the minimum of σ_A is equal to σ_D, (equation 58), the requirement of a minimum value for the numerator gives, with allowance for equation (15):

$$(x_A)^* = \frac{R_x}{J_x} \equiv x_M \tag{146}$$

i.e., the Wagner stress:

$$(\sigma_A)^* \equiv \sigma_M = \frac{G\ J_T + \frac{\pi^2}{l^2}\ E\left(C - \frac{R_x{}^2}{J_x}\right)}{J_p + F\ x_M{}^2} = \frac{G\ J_T + \frac{\pi^2}{l^2}\ E\ C^*}{J_p{}^*} \tag{147}$$

This gives the relation:

$$C^* = C - \frac{R_x{}^2}{J_x} \tag{148}$$

valid for symmetry with the x axis, which can be employed in the numerical determination of C^*.

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11) Formulate: $\delta\left[\int_F w^{*2}\ dF - \frac{1}{F}\left(\int_F w^*\ dF\right)^2\right] = 0$. Then the introduction of equation (14), representing a purely geometrical relation, gives the equations (10) defining the center of shear.

On the other hand, a much more comprehensive view of the relation existing between the Wagner stress σ_M and the actual twisting-failure stress than is afforded from equations (136) to (138) is obtained from equation (134b) if η_M is expressed with φ:

$$\eta_M = \tilde{x}_M \varphi \tag{149}$$

Observing that

$$\tilde{x}_M = x_M - x_D \tag{150}$$

the conventional sine formula gives the following relation:

$$\sigma_D = \frac{G\ J_T + \frac{\pi^2}{l^2}\ E\ C^*}{J_T + F\ x_M\ x_D} \tag{151}$$

For the numerical calculation of σ_D, this equation is not expedient, as it would first require the calculation of the distance x_D between centroid and center of rotation from the quadratic equation (130). The significance of this relation lies in the convenience of comparison with equation (147) for σ_M. Replacing x_D with the shear center distance $x_M = \frac{R_x}{J_x}$ in the left-hand side of equation (130), it becomes negative because, observing equation (148), we obtain the following inequation:

$$- \frac{C^* + \frac{1}{\pi^2}\frac{G}{E}\ J_T\ l^2}{J_x} < 0 \tag{152}$$

x values which in amount are greater than the roots of equation (130) make, on the other hand, the left-hand side positive, giving a picture as shown in figure 10. x_M accordingly always lies between the two roots x_D, of which one is positive, the other negative (their product is given through $- J_p/F$!) according to equation (130). To obtain the lower critical stress σ_D, it is necessary to insert the value having the same prefix as x_M in equation (151). But this x_D value being always greater in amount than x_M according to the foregoing, it follows that $\sigma_D < \sigma_M$.

For great $J_T l^2$ the distances x_D and x_M and consequently, σ_D and σ_M differ very considerably, according to figure 10. The difference becomes less as the wall thicknesses or lengths become less. But in the extreme case of $J_T l^2 = 0$, they disappear for angle and T sections only, because $C^* = 0$ for these sections only, so that then the equal sign takes the place of the $<$ sign for $J_T l^2 = 0$ in (152). For all other (not double symmetrical or point symmetrical) sections, there always remains a finite distance between center of shear and center of rotation and consequently, a difference between σ_M and σ_D (see also equations (139) and (140)) even if $s = 0$ or $l = 0$.

7. SUPPLEMENTARY NOTE FOR THE CASE IN WHICH l and s ARE VERY SMALL

For very small column length and wall thickness the theory advanced here fails because then the twist is accompanied by buckling of the walls and the cross-sectional shape is, as a result, altered. The twisting-failure stresses as computed with equations (58) and (99), respectively, would in that case be no more than a very rough approximation under certain circumstances, and a mathematical treatment of the existing conditions is quite difficult. For curved-section center line, for example, it would present a complicated shell problem involving other than the usually allowed-for "tensile stresses" $\tilde{\sigma}$, "flexural stresses" $\tilde{\tilde{\sigma}}$ linearly distributed over the wall thickness. Besides, the immediate effect of the shearing stresses $\tilde{\tau}$ on the strains would in most cases be no longer negligible, as they are in noway always small on account of the short column length (shell length). They are, therefore, in general, no longer computable with (30) from the equilibrium conditions (21) and (120).

For sections built up from a few straight pieces, the solution may perhaps be somewhat easier, but here also, the argument made for $\tilde{\tau}$, holds true. On the other hand, allowance for the bending stresses $\tilde{\tilde{\sigma}}$ linearly distributed over the wall thickness is made easier, since it is possible to apply simple formulas for the change of cross-sectional shape at buckling. The simplest case of this kind is that of a cruciform section. Here, even for very little wall thickness (in first approximation) the cross-sectional shape is preserved and the twisting of the cross sections about the centroid (double symmetry) is identical with buckling of the walls after one single half-wave, the walls being visualized as plates pin-ended at one side and free at the other. The "tensile stresses" $\tilde{\sigma}$, constant over the wall thickness, disappear in this case because $\tilde{C} = C = C^* = 0$. If b denotes the leg length (width of plates), formula (48), as well as Wagner's conformable formula (77) (S = M!), gives:

$$\sigma_D = \frac{G\ J_T}{J_p} = \frac{E}{2(1+\mu)}\ \frac{s^2}{b^2} \tag{153}$$

The twisting-failure stress or buckling stress σ_D is accordingly unrelated to the column length. But this result is valid only if the length does not fall below a certain amount. Because, as the column length becomes less, the hitherto neglected "flexural stresses" $\tilde{\sigma}$ linearly distributed over the wall thickness, become of ever-increasing significance and result here in a rise of critical compressive stress. The complete Wagner formula (see footnote, p. 32) allows for these bending stresses $\tilde{\tilde{\sigma}}$ by adding the term C_{bdn}, related to the transverse warping, to the term $C_{bdu} \equiv C^*$, connected with the circumferential warping and tensile stresses $\tilde{\sigma}$, respectively. Here $C_{bdu} = 0$ and $C_{bdn} = \left(\frac{1}{9}\right) b^3\ s^3$; and (153) is replaced by the more exact values:

$$\sigma_{Wagner} = \left(\frac{1}{2(1+\mu)} + \frac{\pi^2}{12}\ \frac{b^2}{l^2}\right) E\ \frac{s^2}{b^2} \tag{154}$$

This equation is, up to a factor $1 : (1 - \mu^2)$ in the correction term, in agreement with the approximate term originally obtained by Timoschenko (reference 5) on the basis of the Ritz method for buckling of plates under the previously cited boundary conditions. It is seen that the improvement through C_{bdn} for $\frac{l}{b} > 10$ is already insignificant (< 2 percent).

According to Wagner, the same formula (154) would be used on an angle section of leg length b, but this properly would be permissible only when the axis of rotation

D passed through the point of intersection of the two legs. And this really never happens, as shown at the end of the preceding chapter (unless it is intentionally effected through guides, etc.), while on the other hand, the error is so much less as wall thickness and length are less.[12] For greater ratio l/b, the correction given through $\left(\frac{\pi^2}{12}\right)\left(\frac{b}{l}\right)^2$ is meaningless, because then, properly, it would require a correction toward the other side, which allows for the fact that center of shear and center of rotation are not coincident. Even so, the error is less than 11 percent despite the great value $|x_M|/i_p{}^* = 0.61$, as can be readily proved, because at fairly small ratio l/b, Euler buckling already takes place in direction of the axis of symmetry $\left(\text{at } \left(\frac{s}{b}\right)^2 \left(\frac{l}{b}\right)^2 = 1.07\right)$.

Similar conditions prevail on the uneven leg-angle section and on the T section, where C^* likewise = 0. But for all other sections, allowance would have to be made for the change in cross-sectional shape induced by buckling. Every section therefore presents a new problem which is probably solvable for the simpler cases only. The actual critical stresses are lower, rather than higher, compared to the twisting-failure stresses computed according to (58) or (99).

8. PRACTICAL DETERMINATION OF THE VALUES R_y, R_x, and C

The starting point for computing the warping moments R_y, R_x and of the warping strength C dependent on the cross-sectional shape only, is formed, as seen from equations (37) to (39) by the unit warping w referred to the centroid which, according to figure 1, can be determined as areas. For sections built up from straight pieces, the calculation becomes fairly simple because the w are in sections, linear functions of the circumferential coordinate u, as well as the coordinates x, y of any point of the section center line. Then the integrals for R_y, R_x, and C can be computed exactly from the following formulas, in which ψ and ω denote any linear function of

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[12] In contrast with the doubly symmetrical cruciform section, $\tilde{\sigma}$ and C disappear on the angle section only in the extreme case.

u at interval $u_{k-1} \leqq u \leqq u$:

$$\int_{u_{k-1}}^{u_k} \psi(u)\,\omega(u)\,du = \frac{\Delta u_k}{6}\left[\psi_{k-1}\,(2\omega_{k-1} + \omega_k) + \psi_k\,(2\omega_k + \omega_{k-1})\right]$$

$$= \frac{\Delta u_k}{6}\left[\omega_{k-1}\,(2\psi_{k-1} + \psi_k) + \omega_k\,(2\psi_k + \psi_{k-1})\right] \tag{155}$$

$$\int_{u_{k-1}}^{u_k} \psi^2(u)\,du = \frac{\Delta u_k}{3}\left[\psi_{k-1}^2 + \psi_{k-1}\,\psi_k + \psi_k^2\right] \tag{156}$$

These formulas, whose calculation is best effected by tabulation can, of course, be employed also for ascertaining the inertial and centrifugal moments. They are also applicable for partially or completely curved center line if it is replaced by a set of straights.[13)]

Quite often it will be of advantage to apply, instead of the w referred to S other unit warping $\overline{w}$ referred to any favorably located reference point O. With the equation (fig. 2):

$$\overline{w} = w + \overline{x}_s\,y - \overline{y}_s\,x + K \tag{157}$$

a simple calculus gives the following relations:

$$R_y = \int_F x\,w\,dF = \overline{R}_{\overline{y}} + \overline{y}_s\,J_y - \overline{x}_s\left(J_{xy} + \int_F \overline{w}\,dF\right) \tag{158}$$

$$R_x = \int_F y\,w\,dF = \overline{R}_{\overline{x}} - \overline{x}_s\,J_x + \overline{y}_s\left(J_{xy} - \int_F \overline{w}\,dF\right) \tag{159}$$

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13) For the usual section forms, a tabulation of the numerical values for R_x, R_y, C would be suitable similarly as for the cross-sectional area and the inertia moments.

$$C = \int_F w^2 \, dF - \frac{1}{F}\left(\int_F w \, dF\right)^2 = \overline{C} + \overline{x}_s^2 \, J_x -$$

$$- 2\overline{x}_s \, \overline{y}_s \, J_{xy} + \overline{y}_s^2 \, J_y - 2\overline{x}_s \, \overline{R}_{\overline{x}} + 2\overline{y}_s \, \overline{R}_{\overline{y}} \qquad (160)$$

Hereby

$$\overline{R}_{\overline{y}} = \int_F \overline{x} \, \overline{w} \, dF, \qquad \overline{R}_{\overline{x}} = \int_F \overline{y} \, \overline{w} \, dF \qquad (161), (162)$$

and, corresponding to (107),

$$\overline{C} = \int_F \overline{w}^2 \, dF - \frac{1}{F}\left(\int_F \overline{w} \, dF\right)^2$$

The application of these formulas to a channel section will serve as illustration (fig. 11). Owing to the symmetry with the x axis, it is $R_y = 0$. For computing R_x and C, it is best to use the coordinate system $\overline{x}, \overline{y}$ parallel to x, y that is placed through the intersection O of section center line and axis of symmetry. Observing the prefixes for r_t and $\overline{r}_t$ as given in section 2, it is seen that, when passing from point 2' to point 2, $\overline{r}_t$ has the positive value $a/2$ for both flanges and value 0 for the web. The coordination of the unit warping $w_0 = 0$ to point O, is followed according to equation (101)

$$\overline{w} = \overline{w}_0 + \int_0^u \overline{r}_t \, du$$

by the unit warping antisymmetrical to the x axis:

$$\overline{w}_{2'} = -\frac{ab}{2}, \quad \overline{w}_{1'} = 0, \quad \overline{w}_0 = 0, \quad \overline{w}_1 = 0, \quad \overline{w}_2 = +\frac{ab}{2} \qquad (163)$$

whence

$$\int_F \overline{w}\, dF = 0$$ [14)]

Then equations (155) and (156) give, in simple fashion:

$$\overline{R}_{\overline{x}} = -\frac{1}{4}\, s\, a^2\, b^2 \qquad (164)$$

and

$$\overline{C} = +\frac{1}{6}\, s\, a^2\, b^3 \qquad (165)$$

It further is:

$$\overline{x}_s = b^2 : (a + 2b) \qquad (166)$$

and

$$J_x = \frac{1}{12}\, s\, a^2\, (a + 6b) \qquad (167)$$

The final results, according to equations (159) and (160) are the expressions:

$$R_x = \overline{R}_{\overline{x}} - \overline{x}_s\, J_x = -\frac{s\, a^2\, b^2}{3}\, \frac{a + 3b}{a + 2b} \qquad (168)$$

$$C = \overline{C} + \overline{x}_s{}^2\, J_x - 2\overline{x}_s\, \overline{R}_{\overline{x}} = \frac{s\, a^2\, b^3}{12}\, \frac{2a^2 + 15ab + 26b^2}{(a + 2b)^2} \qquad (169)$$

previously employed in the example of section 3.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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14) Choosing the opposite sense of rotation changes the prefix of $\overline{r}_t$, but not of $\overline{w}$.

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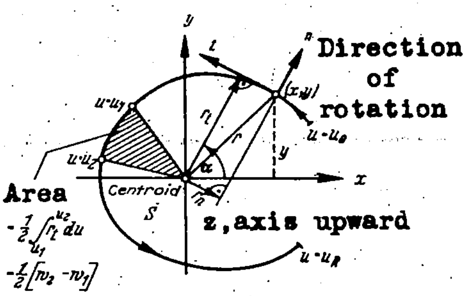

Figure 1.- Section center line of an open section with notation.

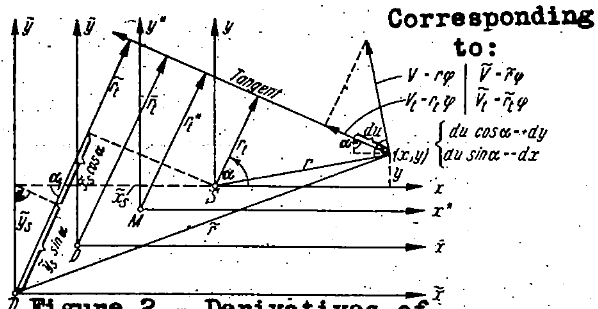

Figure 2.- Derivatives of the unit warping $w, w^*, \bar{w}, \tilde{w}$.

x-y-System through centroid S; $w = w_0 + \int_0^u r_t \, du$

x*-y*-System through center of shear M; $w^* = w^*_0 + \int_0^u r_t^* \, du$

$\bar{x}$-$\bar{y}$-System through any reference point O; $\bar{w} = \bar{w}_0 + \int_0^u \bar{r}_t \, du$

$\tilde{x}$-$\tilde{y}$-System through actual center of rotation D; $\tilde{w} = \tilde{w}_0 + \int_0^u \tilde{r}_t \, du$

$$\tilde{r}_t = r_t + \tilde{x}_S \cos\alpha + \tilde{y}_S \sin\alpha$$
$$\tilde{r}_t = \bar{r}_t + \tilde{x}_O \cos\alpha + \tilde{y}_O \sin\alpha$$
$$\bar{r}_t = r_t + \bar{x}_S \cos\alpha + \bar{y}_S \sin\alpha$$
$$\bar{r}_t = r_t^* + \bar{x}_M \cos\alpha + \bar{y}_M \sin\alpha.$$

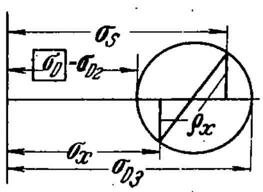

Figure 3.- Determination of the twisting failure stresses σ_{D_2} and σ_{D_3} from σ_S, σ_x, ρ_x by means of Mohr's circle.

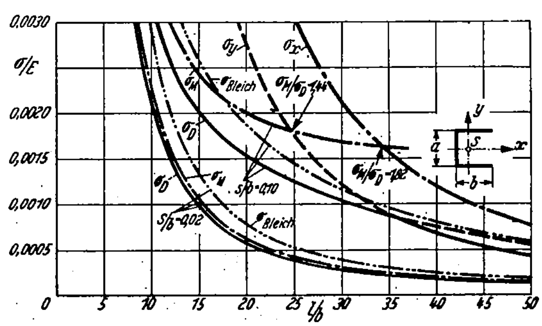

Figure 4.- Twisting failure stress σ_D against length l and wall thickness s of channel sections with a/b = 1.

Euler stresses σ_x and σ_y according to Eq. (73),(74), twisting failure stress σ_D according to Eq.(58) by means of Eq. (73),(75),(76), Wagner stress σ_M according to Eq.(78), Bleich stress σ_{Bleich} according to Eq.(79).

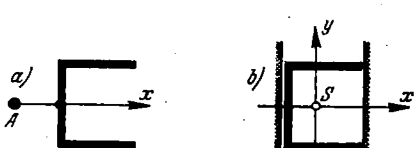

Figure 6.- Channel sections with guides(enforced axis of rotation A).

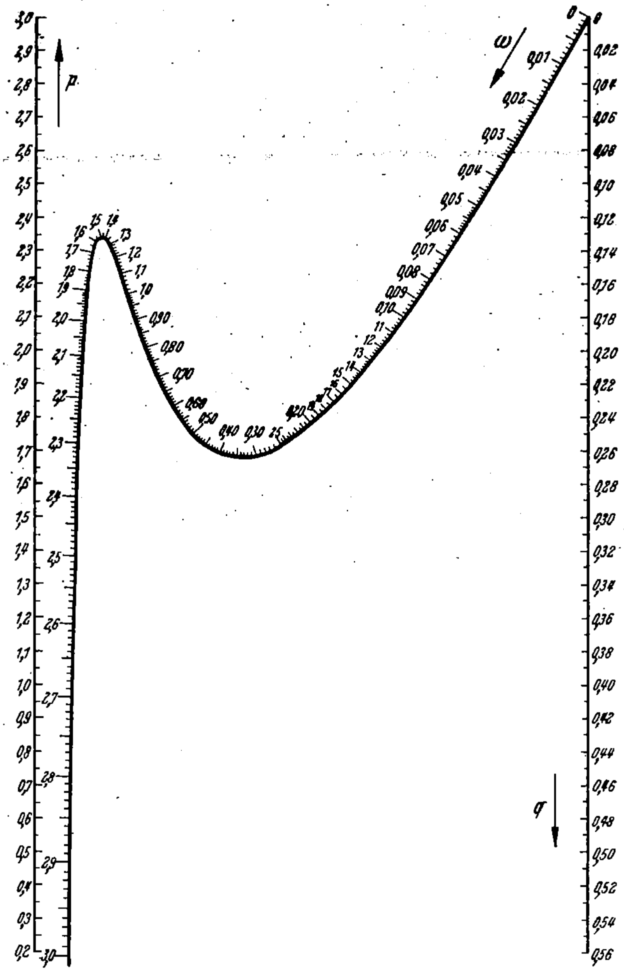

Figure 5.- Chart for determining the twisting failure stress $\sigma_D = \omega\sigma_m$ from the cubic equation

$$\omega^3 - 3\omega^2 + p\omega - q = 0.$$

$$\sigma_m = \frac{\sigma_S + \sigma_x + \sigma_y}{3}$$

$$p = \frac{\sigma_S\sigma_x + \sigma_S\sigma_y + \sigma_x\sigma_y - \varrho_{xy}^2 - \varrho_x^2 - \varrho_y^2}{\sigma_m^2}$$

$$q = \frac{\sigma_S\sigma_x\sigma_y + 2\varrho_{xy}\varrho_x\varrho_y - \varrho_{xy}^2\sigma_S - \varrho_y^2\sigma_x - \varrho_x^2\sigma_y}{\sigma_m^3}.$$

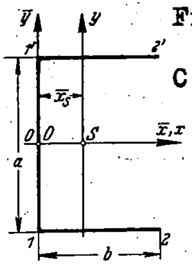

Figure 11.- Solution of R_x and C on U-channel section.

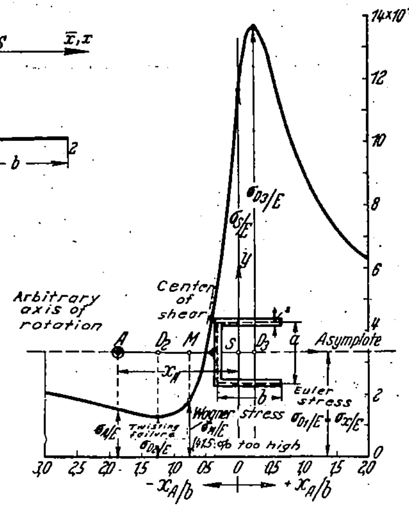

Figure 7.- Twisting failure stress σ_A with enforced axis of rotation A for a channel section with a/b = 1, l/b = 24, s/b = 0.1(Fig.4)

$$\sigma_A = \frac{G\,J_T + \frac{\pi^2}{l^2}E\,(C - 2\,x_A\,R_x + x_A^2\,J_x)}{J_p + F\,x_A^2}$$

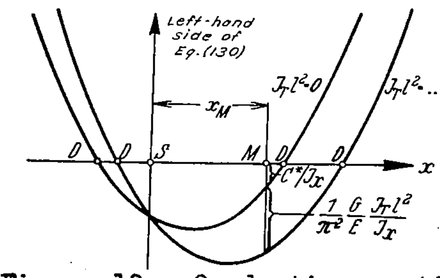

Figure 10.- Quadratic equation (130) for x_D .

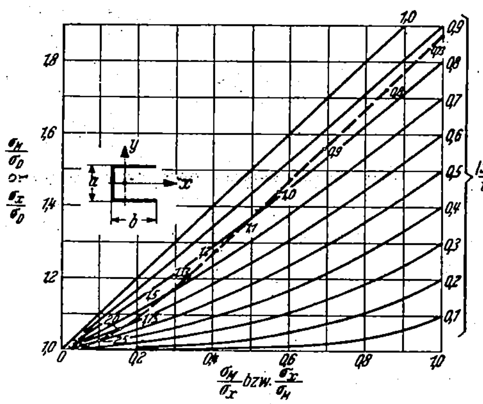

Figure 8.- Curves for computing twisting failure stress σ_D from Wagner's stress σ_M and Euler's stress σ_x for open sections with symmetry to the x axis. x_M/i^*_p for channel sections from Fig. 9; for other sections compute with Eq.(15). ---- curve for channel sections in the extreme case σ_M σ_y. For channel sections with a/b > 0.73, where $\sigma_y < \sigma_x$,it gives the maximum values of σ_M/σ_D.(Digits denote length-width ratio a/b).

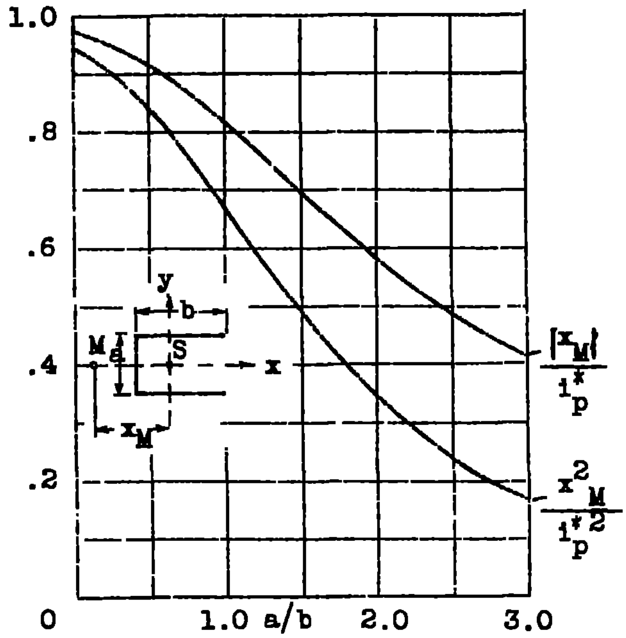

Figure 9.- x^2_M/i_p^{*2} and $|x_M|/i_p^*$ values of channel sections.

$$\frac{x_M^2}{i_p^{*2}} = \frac{196\left(3 + \frac{a}{b}\right)^2}{\left(2 + \frac{a}{b}\right)\left(936 + 276\frac{a}{b} + 224\left(\frac{a}{b}\right)^2 + 108\left(\frac{a}{b}\right)^3 + 18\left(\frac{a}{b}\right)^4 + \left(\frac{a}{b}\right)^5\right)},$$